S.SINHA COLLEGE AURANGABAD BIHAR

P.G NOTES

By Assistant Prof.SHIVAM KUMAR S.SINHA COLLEGE
AURANGABAD

TOPIC NAME:- FUZZY RELATIONS

CLASS :- M.sc(MATHEMATICS)

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Union and Intersection of two fuzzy relations:-

* **Union of fuzzy relations:-** Let R and Z be two fuzzy relations in the same product space, then the union of R and Z is denoted by R∪Z and is defined as

$$R \cup Z = \{[(x,y), \mu_{R \cup Z}(x,y)] : (x,y) \in X \times Z\}.$$

where $\mu_{R \cup Z}(x,y) = \max\{\mu_R(x,y), \mu_Z(x,y) : (x,y) \in X \times Z\}$.

Eg:- If the fuzzy relations R and Z are given by

$$R = \begin{array}{c c} & \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{pmatrix} .8 & 1 & .1 & .7 \\ 0 & .8 & 0 & 0 \\ .9 & 1 & .7 & .8 \end{pmatrix} \end{array} \text{ and } Z = \begin{array}{c c} & \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{pmatrix} .5 & .3 & 0 & 0 \\ .7 & .2 & .5 & 0 \\ 0 & 1 & .3 & .8 \end{pmatrix} \end{array}$$

$$\therefore R \cup Z = \begin{array}{c c} & \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{pmatrix} .8 & 1 & .1 & .7 \\ .7 & .8 & .5 & 0 \\ .9 & 1 & .7 & .8 \end{pmatrix} \end{array}$$

* **Intersection of fuzzy relations:-** Let R and Z be two fuzzy relations in the same product space then the intersection of R and Z is denoted by R∩Z and is defined as

$$R \cap Z = \{[(x,y), \mu_{R \cap Z}(x,y)] : (x,y) \in X \times Z\}.$$

Where $\mu_{R \cap Z}(x,y) = \min\{\mu_R(x,y), \mu_Z(x,y) : (x,y) \in X \times Z\}$.

Eg:- If the fuzzy relations R and Z are given by

$$R = \begin{array}{c c} & \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{pmatrix} .8 & 1 & .1 & .7 \\ 0 & .8 & 0 & 0 \\ .9 & 1 & .7 & .8 \end{pmatrix} \end{array} \text{ and } Z = \begin{array}{c c} & \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{pmatrix} .5 & .3 & 0 & 0 \\ .7 & .2 & .5 & 0 \\ 0 & 1 & .3 & .8 \end{pmatrix} \end{array}$$

$$\therefore R \cap Z = \begin{array}{c c} & \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{pmatrix} .5 & .3 & 0 & 0 \\ 0 & .2 & 0 & 0 \\ 0 & 1 & .3 & .8 \end{pmatrix} \end{array}$$

Ans.

* Fuzzy Equations :-

Fuzzy equation is one of the important area of fuzzy set theory, and fuzzy numbers. These are the equations in which the coefficient and unknowns are fuzzy numbers, and formulas are constructed by operation of fuzzy arithmetic.

Basically, we have two types of equations

$$A + X = B$$

and $$A \cdot X = B.$$

(i) Equation of the type $A + X = B$.

The difficulty in solving this fuzzy equation is caused by the fact that $X = B - A$ is not the solution.

We consider two closed intervals $A = [a_1, a_2]$ and $B = [b_1, b_2]$ which may be viewed as special fuzzy numbers.

Then, $B - A = [b_1, b_2] - [a_1, a_2] = [b_1 - a_2, b_2 - a_1]$

and
$$\begin{aligned} A + B - A &= [a_1, a_2] + [b_1 - a_2, b_2 - a_1] \\ &= [a_1 + b_1 - a_2, a_2 + b_2 - a_1] \\ &\neq [b_1, b_2] \end{aligned}$$

$\Rightarrow A + B - A \neq B.$

Whenever $a_1 \neq a_2$.

$\therefore X = B - A$ is not the solution of the eqn.

Let $X = [x_1, x_2]$ then $A + X = B$.

$\Rightarrow [a_1, a_2] + [x_1, x_2] = [b_1, b_2]$

$\Rightarrow [a_1 + x_1, a_2 + x_2] = [b_1, b_2]$

$\Rightarrow a_1 + x_1 = b_1$ and $a_2 + x_2 = b_2$.

$\Rightarrow x_1 = b_1 - a_1$ and $x_2 = b_2 - a_2$.

Since X must be an interval, it is required that $x_1 \leq x_2$. i.e. the equation $A + X = B$ has a solution iff $b_1 - a_1 \leq b_2 - a_2$.

Relational Join :-

Let $P = [p_{ik}]$ and $Q = [q_{kj}]$ and $R = [r_{ij}]$ be the membership matrices of binary relations s.t. $R = P \circ Q$ then

$[r_{ij}] = [p_{ik}] \circ [q_{kj}]$, where $r_{ij} = \max_{k} \min[p_{ik}, q_{kj}]$

Eg:- Let $P = \begin{bmatrix} .3 & .5 & .8 \\ 0 & .7 & 1 \\ .4 & .6 & .5 \end{bmatrix}$, $Q = \begin{bmatrix} .9 & .5 & .7 & .7 \\ .3 & .2 & 0 & .9 \\ 1 & 0 & .5 & .5 \end{bmatrix}$

then by using $r_{11} = \max[\min[p_{11}, q_{11}], \min[p_{12}, q_{21}], \min[p_{13}, q_{31}]]$

$$\therefore P \circ Q = \begin{bmatrix} .3 & .5 & .8 \\ 0 & .7 & 1 \\ .4 & .6 & .5 \end{bmatrix} \circ \begin{bmatrix} .9 & .5 & .7 & .7 \\ .3 & .2 & 0 & .9 \\ 1 & 0 & .5 & .5 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \end{bmatrix}$$

where r_{ij} can be found as follows:-

$$\begin{aligned} r_{11} &= \max[\min(p_{11}, q_{11}), \min(p_{12}, q_{21}), \min(p_{13}, q_{31})] \\ &= \max[\min(0.3, 0.9), \min(0.5, 0.3), \min(0.8, 1)] \\ &= \max[0.3, 0.3, 0.8] \\ &= 0.8 \end{aligned}$$

$$\begin{aligned} r_{12} &= \max[\min(p_{11}, q_{12}), \min(p_{12}, q_{22}), \min(p_{13}, q_{32})] \\ &= \max[\min(0.3, 0.5), \min(0.5, 0.2), \min(0.8, 0)] \\ &= \max[0.3, 0.2, 0] \\ &= 0.3 \end{aligned}$$

Similarly, we can find; $r_{13} = 0.5$, $r_{14} = 0.5$, $r_{21} = 1$, $r_{22} = 0.2$, $r_{23} = 0.5$, $r_{24} = 0.7$, $r_{31} = 0.5$, $r_{32} = 0.4$, $r_{33} = 0.5$, $r_{34} = 0.6$

$$\text{thus } P \circ Q = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \end{bmatrix} = \begin{bmatrix} 0.8 & 0.3 & 0.5 & 0.5 \\ 1 & 0.2 & 0.5 & 0.7 \\ 0.5 & 0.4 & 0.5 & 0.6 \end{bmatrix}$$

Ans.

Fuzzy Relations:-

Let $X, Y \subseteq R$ be universal set, then $R = \{(x,y), \mu R(x,y)) : (x,y) \in X \times Y\}$ is called a fuzzy relation from X to Y.

Ex:- Let $X = Y = R$. i.e set of real numbers and A = 'considerably larger than'.

The membership function of the fuzzy relation which is of course a fuzzy set on $X \times Y$ can be defined as follows:-

$$\mu A(x,y) = \begin{cases} 0 , & x \leq y \\ (x-y)/10y , & y \leq x \leq 11y \\ 1 , & x \geqslant 11y . \end{cases}$$

or

$$\mu A(x,y) = \begin{cases} 0, & x \leq y \\ [1+(y-x)^{-1}]^{-1}, & x > y . \end{cases}$$

Fuzzy relations can also be defined by matrices.

Let A = 'x considerably larger than y'. then

	y_1	y_2	y_3	y_4
x_1	·8	1	·1	·7
x_2	0	·8	0	0
x_3	·9	1	·7	·8

and B = "y is very close to x". then

	y_1	y_2	y_3	y_4
x_1	·4	0	·9	·6
x_2	·9	·4	·5	·7
x_3	·3	0	·8	·5

Thus we can find the generalization of definition:-

Let $X, Y \subseteq R$ and

$X = \{(x, \mu X(x)) : x \in X\}$; $Y = \{(y, \mu Y(y)) : y \in Y\}$, be two fuzzy sets, then fuzzy relation $R(X,Y)$ is defined as follows:-

$R \equiv R(X,Y) = \{(x,y), \mu_R(x,y)) : (x,y) \in X \times Y\}$ is a fuzzy relation on X and Y if

$\mu R(x,y) \leq \mu X(x) , \forall (x,y) \in X \times Y.$

$\mu R(x,y) \leq \mu Y(y) , \forall (x,y) \in X \times Y.$

or $\mu R(x,y) \leq \min(\mu X(x), \mu Y(y))$.

Fuzzy Relation Equations:-

We know that the composition of two binary relations $P(X,Y)$ and $Q(Y,Z)$ can be defined in terms of an operation on the membership matrices P and Q. This operation involves exactly the same combinations of matrix entries as in the regular matrix multiplication. For matrix multiplication and addition we can use the alternative operations. i.e fuzzy set intersection and union respectively. In the max-min composition, the multiplication and additions are replaced with the min and max operations respectively.

The max-min form of composition is not only viewed as the fundamental composition of fuzzy relations, but it also the form that has been used in various applications.

Let $P(X,Y)$, $Q(Y,Z)$ and $R(X,Z)$ be three binary relations defined on the sets X, Y and Z respectively.

Also, let the membership matrices P, Q and R denoted by $P=[p_{ij}]$, $Q=[q_{jk}]$, $R=[r_{ik}]$ respectively.

Where $p_{ij}=P(x_i,y_j)$, $q_{jk}=Q(y_j,z_k)$, $r_{ik}=R(x_i,z_k)$.

It is to be noted that all entries in the matrices P, Q and R are real numbers lying in the unit interval $[0,1]$.

Let us assume that $P \circ Q = R$.

Where $\circ$ denotes the max-min composition.

Theorem:-

For any fuzzy relation R on X^2, the fuzzy relation $R_{T(i)} = \bigcup_{n=1}^{\infty} R^{(n)}$ is the smallest i-transitive closure of R.

Proof:- $R_{T(i)} \overset{i}{\circ} R_{T(i)} = \left(\bigcup_{n=1}^{\infty} R^{n}\right) \overset{i}{\circ} \left(\bigcup_{m=1}^{\infty} R^{m}\right)$

$$= \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \left(R^{(n)} \overset{i}{\circ} R^{(m)}\right) = \bigcup_{n,m=1}^{\infty} R^{(n+m)}$$

$$\subseteq \bigcup_{k=1}^{\infty} R^{(k)}$$

$$= R_{T(i)}$$

means that $R_{T(i)}$ is i-transitive.

Consider now a fuzzy relation S that is i-transitive and contain R $(R \subseteq S)$, then

$$R^{(2)} = R \overset{i}{\circ} R \subseteq S \overset{i}{\circ} S \subseteq S.$$

and moreover, if $R^{(n)} \subseteq S$, then

$$R^{(n+1)} = R \overset{i}{\circ} R^{(n)} \subseteq S \overset{i}{\circ} S \subseteq S.$$

Hence, $R^{(k)} \subseteq S$, for any $k \in N$ and therefore

$$R_{T(i)} = \bigcup_{k=1}^{\infty} R^{(k)} \subseteq S.$$

i.e $R_{T(i)}$ is the smallest i-transitive fuzzy relation containing R.

Compatability Relation:- Binary relation $R(X,Y)$, i.e. reflexive and symmetric is called compatible or tolerance relation. If R is a fuzzy compatible, then comatibility classes are defined in terms of a specified membership degree α. An α-compatible class is a set A of X, s.t.

$R(x,y) \geq \alpha \;\; \forall x,y \in A.$

Theorem:- Let R be a reflexive fuzzy relation on X^2, where $|X| = n \geq 2$, then $R_{T(i)} = R^{(n-1)}$.

Proof:- Since R is reflexive, $E \subseteq R$ and

$$R = E \overset{i}{\circ} R \subseteq R \overset{i}{\circ} R = R^{(2)}.$$

Thus $R^{(m)} \subseteq R^{(m+1)}$ for any $m \in N$.

Now, we prove that $R^{(n-1)} = R^{(n)}$ for any $x, y \in X$, if $x = y$.

then $R^{(n)}(x, y) = \sup i[R(x, z_1), R(z_1, z_2), \ldots, R(z_{n-1}, y)]$

Since $|X| = n$ then sequence $x = z_0, z_1, \ldots, z_{n-1}, z_n = y$ of $n+1$ elements must contain at least two equal elements. Assume $z_r = z_s$ where $r < s$ then

$$i[R(x, z_1) \ldots, R(z_{r-1}, z_r) \ldots R(z_s, z_{s+1}) \ldots R(z_{n-1}, y)]$$
$$\leq i[R(x, z_1), \ldots R(z_{r-1}, z_r), R(z_s, z_{s+1}) \ldots R(z_{n-1}, y)]$$
$$\leq R^{(k)}(x, y)$$
$$\leq R^{(n-1)}(x, y)$$

Hence $R^{(n)}(x, y) \leq R^{(n-1)}(x, y)$ for any $x, y \in X$.

$\Rightarrow R^{(n)} \subseteq R^{(n-1)}$.

$\Rightarrow R_{T(i)} = R^{(n-1)}$.

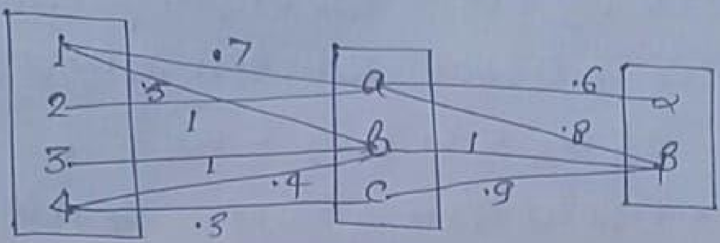
Proved.

Max-Min compositions:-

Max-Min composition is obtained by aggregating appropriate elements of the corresponding Join by the Max operator s.t.

$$(P\circ Q)(x,z) = \max_{y}[P*Q](x,y,z)\ ,\ \forall x\in X,\ z\in Z.$$

Eg:- Let $X=\{1,2,3,4\}$, $Y=\{a,b,c\}$, $Z=\{\alpha,\beta\}$ be three fuzzy sets, then a relational Join exists as



$P(X,Y) =$

	a	b	c
1	.7	.5	0
2	1	0	0
3	0	1	0
4	0	.4	.3

; $Q(Y,Z) =$

.6	.8
0	1
0	.9

$$\therefore\ P\circ Q = \begin{bmatrix} .7 & .5 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & .4 & .3 \end{bmatrix} \circ \begin{bmatrix} .6 & .8 \\ 0 & 1 \\ 0 & .9 \end{bmatrix} = \begin{bmatrix} .6 & .7 \\ .6 & .8 \\ 0 & 1 \\ 0 & .4 \end{bmatrix}$$

* Properties of Max-Min composition:-

1. Associativity:- The max-min composition is associative.
   i.e. $(R_1\circ R_2)\circ R_3 = R_1\circ(R_2\circ R_3)$.
2. Reflexivity:- Let R be a fuzzy relation on $X\times X$, then
   (i) R is said to be reflexive if
   $\mu_R(x,x)=1\ \forall x\in X$.
   (ii) R is said to be ϵ reflexive if
   $\mu_R(x,x)\geq \epsilon\ \forall x\in X$.
   (iii) R is said to be weakly reflexive if
   $$\left.\begin{array}{l}\mu_R(x,y)\leq \mu_R(x,x)\\ \mu_R(y,x)\leq \mu_R(x,x)\end{array}\right\}\ \forall x,y\in X.$$

Transitive closure:-

The transitive closure of a crisp relation $R(x,x)$ is the relation which is transitive contain $R(x,x)$ and has the minimum possible element. On the other hand, for fuzzy relation, the transitive closure of a fuzzy relation is the generalization that the elements of the transitive closure have the smallest possible membership grades that still allow the first two components to be met. The transitive closure $R_T(X,X)$ of a relation $R(X,X)$ is determined by simple algorithm that consists of the following three steps:-

(i) $R' = R \cup (R \circ R)$

(ii) If $R' \neq R$, make $R = R'$ and go to (i)

(iii) Stop, $R' = R_T$.

Eg:- Let $R = \begin{bmatrix} .8 & .6 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ .2 & .4 & 0 & 0 \\ 0 & 0 & .9 & .8 \end{bmatrix}$ then

$$R \circ R = \begin{bmatrix} .8 & .6 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ .2 & .4 & 0 & 0 \\ 0 & 0 & .9 & .8 \end{bmatrix} \circ \begin{bmatrix} .8 & .6 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ .2 & .4 & 0 & 0 \\ 0 & 0 & .9 & .8 \end{bmatrix} = \begin{bmatrix} .8 & .6 & .6 & .6 \\ .8 & .6 & .9 & .8 \\ .4 & .2 & .4 & .4 \\ .2 & .4 & .8 & .8 \end{bmatrix}$$

then $R \cup (R \circ R) = \begin{bmatrix} .8 & .6 & .6 & .6 \\ 1 & .6 & 1 & 1 \\ .4 & .4 & .4 & .4 \\ .2 & .4 & .9 & .8 \end{bmatrix} = R'$.

Here $R' \neq R$. So, we again repeat the above step

$$R' \circ R' = \begin{bmatrix} .8 & .6 & .6 & .6 \\ .8 & .6 & .9 & .8 \\ .4 & .4 & .4 & .4 \\ .4 & .4 & .8 & .8 \end{bmatrix} \Rightarrow R' \cup (R' \circ R') = \begin{bmatrix} .8 & .6 & .6 & .6 \\ 1 & .6 & 1 & 1 \\ .4 & .4 & .4 & .4 \\ .4 & .4 & .9 & .8 \end{bmatrix} = R''.$$

$\because R'' \neq R'$, again, we repeat the same step.

$$\therefore R'' \circ R'' = \begin{bmatrix} .8 & .6 & .6 & .6 \\ .8 & .6 & .9 & .8 \\ .4 & .4 & .4 & .4 \\ .4 & .4 & .8 & .8 \end{bmatrix} = R'' \Rightarrow R'' \cup (R'' \circ R'') = R'''.$$

Here $R''' = R''$. Thus $R''' = R_T$.

Q. Show that a fuzzy relation R is called a max-min transitive if $R \circ R \subseteq R$.

Solution:- Let $R = \begin{bmatrix} .2 & 1 & .4 \\ 0 & .6 & .3 \\ 0 & 1 & .3 \end{bmatrix}$

Now, firstly we shall prove that R is transitive.

$\therefore R \circ R = \begin{bmatrix} .2 & 1 & .4 \\ 0 & .6 & .3 \\ 0 & 1 & .3 \end{bmatrix} \circ \begin{bmatrix} .2 & 1 & .4 \\ 0 & .6 & .3 \\ 0 & 1 & .3 \end{bmatrix}$

$= \begin{bmatrix} \max(.2,0,0) & \max(.2,.6,.4) & \max(.2,.3,.3) \\ \max(0,0,0) & \max(0,.6,.3), & \max(0,.3,.3) \\ \max(0,0,0) & \max(0,.6,.3) & \max(0,.3,.3) \end{bmatrix}$

$\therefore R \circ R = \begin{bmatrix} .2 & .6 & .3 \\ 0 & .6 & .3 \\ 0 & .6 & .3 \end{bmatrix}$

Now, $(R \circ R) \cup R = \begin{bmatrix} .2 & .6 & .3 \\ 0 & .6 & .3 \\ 0 & .6 & .3 \end{bmatrix} \cup \begin{bmatrix} .2 & 1 & .4 \\ 0 & .6 & .3 \\ 0 & 1 & .3 \end{bmatrix} = \begin{bmatrix} .2 & 1 & .4 \\ 0 & .6 & .3 \\ 0 & 1 & .3 \end{bmatrix} = R'$

Here $(R \circ R) \cup R = R'$. Thus $R' = R_T$, the closure membership matrix, so R is transitive.

Now, we shall prove that $R \circ R \subseteq R$.

Clearly, we see that $\mu_{R \circ R}(x) \leq \mu_R(x)$

So, $R \circ R \subseteq R$.

Proved.

Fuzzy order relation:- A fuzzy relation which is max-min transitive, reflexive and antisymmetric is called a fuzzy order relation.

- If a relation is perfectly antisymmetric, then relation is called perfect fuzzy relation. It is also called fuzzy partial order relation.
- A total fuzzy order relation or a fuzzy linear ordering is a fuzzy order relation s.t for all $x, y \in X$, $x \neq y$ either $\mu_R(x,y) > 0$ or $\mu_R(y,x) > 0$.

Composition :-

Let $P(X, Y)$ and $Q(Y, Z)$ with a common set Y, then their composition is denoted as $P(X,Y) \circ Q(Y,Z)$ and is defined as $R(X,Z)$ on $X \times Z$ where

$$R(x,z) = [P \circ Q](x,z) = \max_{y \in Y} \min[P(x,y), Q(y,z)]$$

$$\forall x \in X, z \in Z.$$

The above composition which is based on the standard t-norms and t-conorms is often referred as the max-min composition.

For composition,

$$[P(X,Y) \circ Q(Y,Z)]^{-1} = Q^{-1}(Z,Y) \circ P^{-1}(Y,X).$$

and $[P(X,Y) \circ Q(Y,Z)] \circ R(Z,W) =$

$$P(X,Y) \circ [Q(Y,Z) \circ R(Z,W)].$$

thus the standard composition is associative however, the standard composition is not commutative, because $Q(Y,Z) \circ P(X,Y)$ is not well defined, when $X \neq Z$.

Even if $X = Z$ and $Q(Y,Z) \circ P(X,Y)$ are well defined, we have

$$P(X,Y) \circ Q(Y,Z) \neq Q(Y,Z) \circ P(X,Y).$$

* Properties of composition

1. standard composition is associative, i.e $[P(X,Y) \circ Q(Y,Z)] \circ R(Z,W) = P(X,Y) \circ [Q(Y,Z) \circ R(Z,W)]$.
2. The inverse of the standard composition is equal to the reverse composition of inverse relation.
3. standard composition is not commutative.
4. If $P = [p_{i,k}]$, $Q = [q_{kj}]$ and $R = [r_{ij}]$ then $R = P \circ Q$. Where $[r_{ij}] = [p_{ik}] \circ [q_{kj}]$.

Fuzzy Relation Equations (contd):—

$\because P \circ Q = R$ —(i)

Where $\circ$ denotes the max-min composition.

The equation (i) can also be written as

$$\max_j \min(p_{ij}, q_{jk}) = r_{ik} \quad \text{—(ii)}$$

The matrix equation (i) having $n \times s$ simultaneous equation of the form (ii). If two of the components in each of the equations are known and one is unknown, then these equations are known as fuzzy relation equations.

⇒ If matrix P and Q are given and R to be determined from (i) then this is a trivial case. In this case a unique solution exists.

Problem partitioning:- Assume that a pair of matrices R and Q is given. We want to find the set of all particular matrices of the form P that satisfy (i).

let $S(Q,R) = [P : P \circ Q = R]$ —(iii)

denote the set of solutions.

We can partition the given problem in the following form:-

$$P_i \circ Q = r_i \quad \text{—(iv)}$$

Where $P_i = [P_{ij}]$ and $r_i = [r_{jk}]$

Here we observe that P_i, Q and r in (iv) represent respectively a fuzzy set on Y, a fuzzy relation on $S_i(Q,r_i) = [P_i : P_i \circ Q] = r_i$ —(v)

The matrices P in (iii) can be seen as one column matrices $P = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{bmatrix}$ where $P_i \in S_i(Q, r_i)$ $\forall$ $i \in Z$.

Fuzzy Graph:-

We know that any relation between two sets X and Y is called binary relation denoted by R[X,Y].

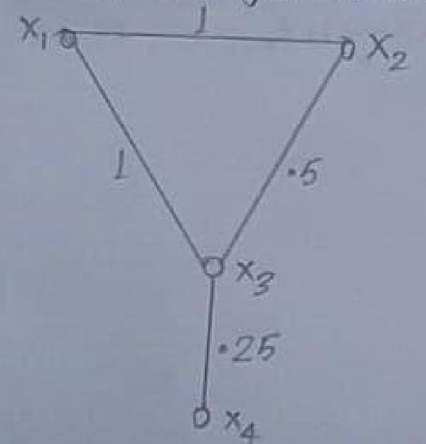
When X ≠ Y then binary relation R(X,Y) is called bipartite graph and when X = Y then binary relation R(X,Y) is called directed graph or digraph.

⇒ Important points for graph:-

(i) $H(x_i, x_j)$ is a fuzzy subgraph of a graph $G(x_i, x_j)$ if $\mu H(x_i, x_j) \subseteq \mu G(x_i, x_j)$ $\forall x_i, x_j \in N \times N$.

(ii) $H(x_i, x_j)$ spans the graph $G(x_i, x_j)$ if the node sets of $H(x_i, x_j)$ and $G(x_i, x_j)$ are equal. i.e if they differ only by their arc weight.

Eg:- Consider the following fuzzy graph:-



(i) Give an example of a spanning graph G.

(ii) Give all paths from x_1 to x_4 and determine their μ length.

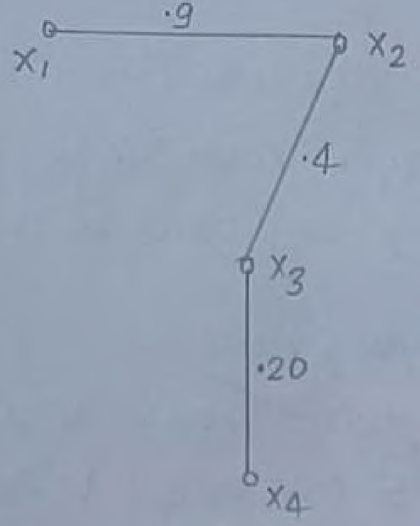
(iii) Is the graph a forest or a tree. If not make it.

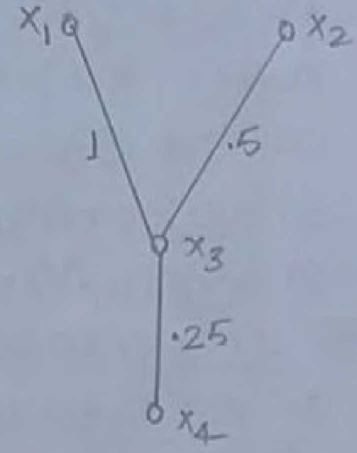
Soln:- (i) It is a spanning graph $H(x_i, x_j)$.
Where $\mu H(x_i, x_j) \leq \mu G(x_i, x_j)$.

μ length ⇒ the μ distance or length $d(x_i, x_j)$ between two nodes (x_i, x_j) is the smallest μ-length of any path from x_i to x_j $\forall x_i, x_j \in G$.

(ii) $d(x_1, x_2) = 1$; $d(x_1, x_3) = 1$; $d(x_2, x_3) = .5$; $d(x_3, x_4) = .25$

$\therefore d(x_1, x_4) = \min\{(1+.5+.25), (1+.25)\} = 1.25$





(iii) It is neither a forest nor a tree.
There is a closed loop, degree of $x_4 = 1$; $\deg(x_3) = 3$. i.e here we see that two edges are of odd degree, so, it is not a tree.